Networks of communities and evolution of cooperation

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In this paper, we propose a new very simple mechanism supporting the emergence of cooperation in a population of memoryless agents playing a prisoner's dilemma game. Each agent belongs to a community and interacts with the agents of its community and with the agents belonging to linked communities. A simple rule governs the dynamics of the system : a community grows (resp decreases) if the average score of its members is superior (resp inferior) to the average score calculated on the whole population. Starting from a random initialisation, the system can evolve towards a majority of cooperators, towards the elimination of cooperators, or towards a situation with periodic evolutions of the populations of cooperators and defectors. The initial presence of clusters of C strategies accounts for the convergence towards cooperative final states. We consider various topologies : Erdős and Rényi random graphs, square lattices and scale-free graphs. Clusters are not as likely to appear in all these topologies, so that there are significant differences between the average frequencies of cooperators associated with each topology. We show that random graphs favours cooperation whereas scale-free graphs tend to inhibit it. The relation between periodic evolutions and topological features is less clear. Nonetheless, we also state the importance of specific C-clusters for the survival of C strategies in periodic oscillations. One major lesson of this paper is that the evolution of cooperation is very sensitive to initial conditions in models with global variables.

1. Introduction

Exploration of the conditions upon which cooperation can emerge has been an active field of investigations in the past years. Various models have been studied. The pioneer work of Axelrod [1981, 1984] shows how cooperation emerges with individuals engaged in iterated interactions. In such interactions, it is worth cooperating to encourage reciprocated cooperation in further encounters. Emergence of cooperation in iterated interactions has been well documented [Boyd & Lorberbaum, 1987; Nowak, 1990; Nowak & Sigmund, 1990;Delahaye & Mathieu, 1992]. Another mechanism known as cliquishness has been proposed [Hruschka & Henrich, 2006]. In this model a set of preferred agents is associated with each agent. An agent is nice (playing cooperation) and tolerant with the agents he prefers and is more likely to retaliate against other agents. Cooperation is then sustainable even with error-prone agents, but agents are no more simple strategies playing the prisoner's dilemma.

Nowak and May [1993;1994a;1994b] developed a model in which memoryless cooperators can survive and invade populations of defecting strategies. The authors have specified a mimetic dynamic : an agent changes its strategy to the strategy of its most successful neighbour. The emergence of cooperation comes from the spatial distributions of strategies and especially from the existence of clusters of C strategies [Szbaó & Tőke, 1997; Schweitzer *et al.*, 2002]. A C strategy with enough C neighbours can be efficient and imitated by its neighbours. In this model, clusters of C strategies are the basic structures which allow for cooperation to survive and grow.

The importance of clusterized structures for the emergence of cooperation was also underlined by Watts [1999] in the case of populations of homogeneous agents who cooperate if a given number of their neighbours cooperate.

We present here another new and simple mechanism for the emergence of cooperation. Our model differs from the Nowak and May model but the strategies we consider are the same as the ones they used.

We consider networks of communities. Communities are sets of strategies. A strategy of one particular community Co interacts with the strategies of Co and with the strategies belonging to communities linked to Co. For each strategy, we compute an average score upon the set of its interactions. The evolution mechanism corresponds to the growth (resp. decrease) of communities which strategies have a high (resp low) average score. Section 2 describes the model and Section 3 studies the kind of evolutions it induces on regular lattices of dimension 2.

As for the dynamics specified by Nowak and May, our dynamic induces very different patterns of evolution when we consider various kinds of networks. Veinstein and Arenzon [2001] considered the model of Nowak and May as a particular case of a more general model where cells can be in three states : C, D or empty. The fact that some sites might be empty is equivalent to the addition of randomness in the structure. The authors showed that cooperation is enhanced with the addition of empty sites. Abramson and Kuperman [2001] explored the model of Nowak and May on small-world structures generated according to the Watts and Strogatz rewiring procedure [Watts & Strogatz, 1998]. They found that for certain values of connectivity and rewiring probability, there are significant increases in the frequency of defectors. We address the question of the impact of the topology in our model in Sec. 4.

The evolutionary mechanism we have specified mainly induces convergence towards two stationary states, one in which cooperation is majoritarian and another in which cooperation disappears. However, some initialisations don't converge. For these initialisations, evolutions of C and D populations are periodic. The frequency of initialisations leading to periodic evolutions depends on the network topology. In Sec. 5, we study periodic phenomena with respect to topological features.

2. The Model

Communities

The agents we consider are memoryless strategies playing the prisoner's dilemma game. Two kinds of agents are possible : cooperators (C) and defectors (D). The following matrix gives the result for each possible interaction :

	C	***D***
C	(3,3)	(5,0)
D	(0,5)	(1,1)

We define a community as a set of strategies. In the rest of the paper, we will only consider sets of C strategies and sets of D strategies, we refer to them as C-communities and D-communities, $type_j \in \{C,D\}$ refers to the kind of strategies on community j. The number of strategies in community j at the t-th generation is e^t_j. Communities are linked and form a graph. We only consider non directed graph here. N denotes the number of communities in the graph. V(j) refers to the set of communities linked to j.

A strategy from the j-th community plays a PD game against the strategies of its community. Its "internal score" is :

$$scoreI^t_j = (e^t_j - 1) * score(type_j, type_j) \quad (1)$$

The strategy plays also against the strategies which belong to the communities in V(j), its "external score" is :

$$scoreE^t_j = \sum_{k \in V(j)} e^t_k * score(type_j, type_k) \quad (2)$$

We define the number of contacts for a strategy of the j-th community as the number of strategies it interacts with. Therefore the number of contacts for a strategy of community j is :

$$nbContacts_j^t=(e_j-1)+\sum_{k\in V(j)} e_k \quad (3)$$

The average score obtained by a strategy of the j-th community is :

$$scoreN_j^t=\frac{scoreI_j^t+scoreE_j^t}{nbContacts_j} \quad (4)$$

At each generation, an average score can be associated with a community. The evolution is then given by :

$$e_j^t=\frac{scoreN_j^{t-1}*e_j^{t-1}}{\sum_{i=1}^{N} scoreN_i^{t-1}*e_i^{t-1}}*\sum_{i=1}^{N} e_i^{t-1} \quad (5)$$

The e_j^t are real values, they are rounded to the closest integer values in order for the populations of communities to be integer values. We see in Sec. 3 that this choice of rounding method has a certain impact on the properties of the cooperative stationary states.

With Eq. (5) the global number of strategies remains constant during evolution. We could assume growing populations but the main results would remain unchanged.

The mechanism is equivalent to a very simple criterion. Let us define global average score as :

$$Q^{t-1}=\frac{\sum_{i=0}^{N} scoreN_i^{t-1}*e_i^{t-1}}{\sum_{i=0}^{N} e_i^{t-1}} \quad (6)$$

using (6), we have :

$$\frac{e_j^t}{e_j^{t-1}}=\frac{scoreN_j}{Q^{t-1}} \quad (7)$$

So the evolutionary mechanism is equivalent to the following criterion : a community grows (resp. decreases) when its average score is superior (resp inferior) to the average score computed on the whole graph. As the global population remains constant with Eq. (5), the dynamic is equivalent to a redistribution of strategies towards the most efficient communities. This model is very simple, the success of a strategy is directly linked to its strength compared to the average strength of other strategies which is a kind of global variable.

In the rest of the paper, we generate a topology and then affect a community to each node. We note p the probability to initialize a node with a C-community, (1-p) being the probability to initialize it with a D-Community. We study both the impact of the kind of graph generated and the impact of p.

A simple example might illustrate the evolutionary mechanism. We consider a C-community of size e_0 and a D-community of size e_1. The two communities are linked.

Let q_0 denote the average score of a strategy C and q_1 denote the average score of a strategy D. N is a total number of strategies, for all generation t :

$$N=e_1^t+e_0^t \quad (8)$$

With these notations we have :

$$e_1^t=\frac{q_1*e_1^{t-1}}{q_1*e_1^{t-1}+q_0*e_0^{t-1}}*N \quad (9)$$

so that :

$$\frac{e_1^t}{e_1^{t-1}}=\frac{q_1*e_1^{t-1}+q_1*e_0^{t-1}}{q_1*e_1^{t-1}+q_0*e_0^{t-1}} \quad (10)$$

The fact that the game is a PD game induces $q_1 > q_0$ for all values e1 and e0, therefore :

$$\frac{e_1^t}{e_1^{t-1}}>1 \quad (11)$$

So, the system converges towards a situation where C strategies disappear. The C-community empties during the evolution.

The result we obtain for two related communities corresponds to a general result on complete graphs. Complete graphs are equivalent to non spatially distributed populations. In non spatially distributed populations, strategies C are eliminated, whether the dynamic is which specified by Nowak and May or the one presented here. This result is a direct consequence of the fact that D is a strictly dominant strategy in the PD game. The limitation of the contacts between communities enables for memoryless cooperative strategies to survive in some configurations. We first analyse the case of square lattices of dimension 2.

3. Emergence of Cooperation with a Regular

Topology

The first results are obtained with square lattices of dimension 10 (N=100) and periodic boundaries. Each community interacts with its eight nearest neighbours. The size of each community is set to 100. With a square lattice topology, the system can be interpreted as a cellular automaton with 2*100*N states per cell.

Three final states are possible here. First, the system converges towards a stationary state with a large majority of cooperators, this is the success of cooperation. Second, the system converges towards the total elimination of C strategies, only D-communities, which all have an average score of 1, remain. This is the dramatic evolution to a generalized state of conflict. The last kind of evolution we observe is periodic : the system doesn't converge and C and D strategies survive in comparable proportions in cyclic variations.

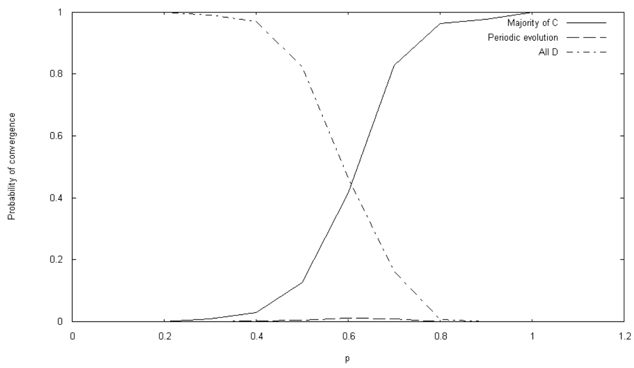

Fig. 3.1 : the probabilities associated with each kind of output starting from a random initialisation. The result are given with respect to the evolution of p, the frequency of C-communities.

There are non-zero probabilities for periodic evolution only for p between 0.4 and 0.7. However, even in this interval, the probabilities for periodic evolutions remain very low. We deal with these cases in Sec. 5. The main result on the graph is that the probability of convergence towards cooperative states increases with p. This comes from the fact that, in our model, emergence of cooperation is related to particular patterns which emergence is favoured by strong p values. We refer to such patterns as protected communities.

We define a protected community as a C-Community which neighbours are also C-communities. For such communities, average score remains constant at 3.

The average score of other communities can be greater than 3 : it is the case for D-communities surrounded by C-communities for example. As the population of C-communities close to D-communities tend to decrease, there is a decrease of the average score of D-communities during evolution. The scores of these D communities will eventually become inferior to 3 and their population will be redistributed towards protected communities if these communities have not been emptied during evolution.

Our simulations showed that the initial presence of protected communities induces a convergence towards a majority of cooperators, with a large part of the population concentrated on protected communities at the end of evolution. From each experimental run from an initial configuration with at least one protected community, we noticed this convergence . In such cases, there remain some D-communities : there are linked to C-neighbours of protected communities. This survival of D strategies is artificial as it depends on the way we obtain integer values from the real values computed with Eq. (5). If we had chosen to obtain integers by truncation of the real values rather than by rounding to the closest integer value, the initial presence of protected communities would have induced final states where the only non-empty communities are protected communities, the whole population begin equally distributed among these communities.

The following development states that it is not possible for protected communities to empty during evolution. We have not yet found a mathematical proof to sustain that protected communities can't be emptied, the development we propose relies on an approximation, it is an heuristic argument.

One can approximate the global average score. The basic idea is to compare this score to 3, the average score for strategies of an protected community.

We use APD to refer to the average

population of a D-community and APC to refer to the average population of a C-community. ASC(k) (resp ASD(k)) is the average score for a strategy of a C-community (resp. D-community) surrounded with k C communities. We have the following approximation :

$$ASD(k) = \frac{5k * APC + (8-k)APD + APD - 1}{APD - 1 + k * APD + (8-k) * APD} \quad (12)$$

$$ASC(k) = \frac{3 * (APC * k + APC - 1)}{APC - 1 + k * APC + (8-k) * APD} \quad (13)$$

If we take SNC(p) (resp SND(p)) as the average score for a C-community (resp D-community), we can state, with X denoting the number of C neighbours for a randomly chosen strategy :

$$SNC(p) = \sum_{(k=0)}^{8} p(X=k) * SNC(k) \quad (14)$$

$$SND(p) = \sum_{(k=0)}^{8} p(X=k) * SND(k) \quad (15)$$

Therefore, we can compute an approximation for the global average score. By computing approximations of the global average score for all possible values of NMC, NMD and p, we concluded that global average score remains inferior to 3.

In the case of square lattices, the population of protected communities can't decrease. The existence of D-communities with average scores superior to 3 is balanced by the existence of C-communities with average scores inferior to 3. As protected communities can't disappear, we are sure that these communities will grow after some generations. Therefore, the existence of protected communities in a regular lattice is a sufficient condition to ensure convergence towards a stationary state with a majority of cooperators. The sufficient condition for the convergence towards a majority of cooperators becomes a sufficient condition for convergence towards a all-C population if we change the way we obtain integer values from Eq. (5).

4. Other Topologies

We have seen a sufficient condition for the emergence of cooperation on a square lattice. We address now the question of the dynamic we specified on more realistic topologies. We consider the evolution of cooperation in the case of scale-free graphs generated with the method described by Barabási and Albert [1999;2002] and in the case of Erdös-Rényi random graphs [Erdős & Rényi, 1959]. The parameter p still denotes the probability for a community to be a C-community.

The method described by Barabási and Albert consists in adding node by node to a random graph with an average degree d. Each added node is connected to d pre-existing nodes of the graph. The probability for choosing a particular pre-existing node increases with the degree of this node. The final result is a graph with an heterogeneous degree distribution and some high connected nodes we refer to as hubs.

We generate graphs with the same cardinality and the same average degree as the square lattice considered in Sec. 3 : cardinality is set to 100 and average degree to 8. We initialize graphs the same way we did in Sec. 3.

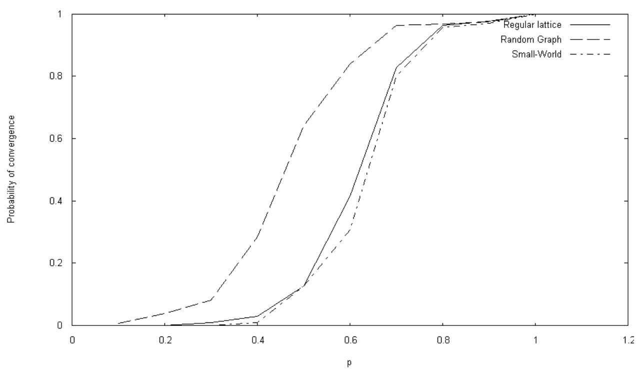

Fig. 4.1 Evolution of the average frequency of cooperators. Each curve corresponds to a different topology. The cardinality of each graph is 100 and its average degree is 8.

Figure 4.1 shows the evolution of the average frequencies of cooperators with respect to p in the three graphs we introduced. Cooperation is favoured by random graphs and not by scale-free graphs here.

We can explain these differences by the probability for generating protected communities associated with each topology. Let

us consider the case of Erdös-Rényi graphs. Let q denote the probability for a randomly chosen community to be an protected community.
We have :

$$q(p)=p*\sum_{k=0}^{N} p_k * p(isolated\ community / d=8) \quad (16)$$

where p_k refers to the probability for a randomly chosen community to be of degree k, p (protected community / d=8) referring to the probability for a community to be protected, its degree being 8.

The degree distribution of a random graph is a Poissonian distribution (a demonstration for this classical result can be found in [Barabási and Albert, 1999]) so that we can compute the values of the q(p) coefficients. q(p) represents the probability to have an protected community, so 1*q(p) is the average frequency of cooperators if we consider that an initial protected community induces convergence towards a state with no defectors. We compared the q(p) values to the empirical average frequencies of cooperators that appear in Fig. 4.1.

Theoretical values fit empirical ones. In the case of scale-free graphs, we ran a similar experience. From a great number of initial configurations, we evaluated the frequencies f of initializations with protected communities. We compared these with the empirical values of average frequencies of cooperators. We found a good fit in this case too. We can therefore conclude that the differences in the average frequencies of cooperators associated with each graph come from the differences between the probability of generating protected communities for each graph. Here again, a mathematical demonstration may exist but has not yet been found. In scale-free graph with heterogeneous degree distributions, the existence of D-hubs prevents the emergence of protected communities for a large number of C-communities. In Erdös-Rényi random graphs, the stronger probabilities of convergence towards a large majority of cooperators come from the existence of some communities with degrees inferior to the average degree and the fact that D-communities can only have a limited interaction range as their degree can't differ too much from the average degree.

To confirm our conclusions, we have tested how sensitive they were to the average degree of the graphs. We have considered the average frequencies of cooperators for the same topologies with average degree of 4 and 12. The main results remained the same.

5. Periodic Configurations

Apart from the two main cases of convergence, some initial states induce periodic oscillations. We give an example of such evolutions in Figs. 5.1 and 5.2. A more detailed evolution for a system with periodic oscillations is given in [Dorat & Delahaye, 2006]

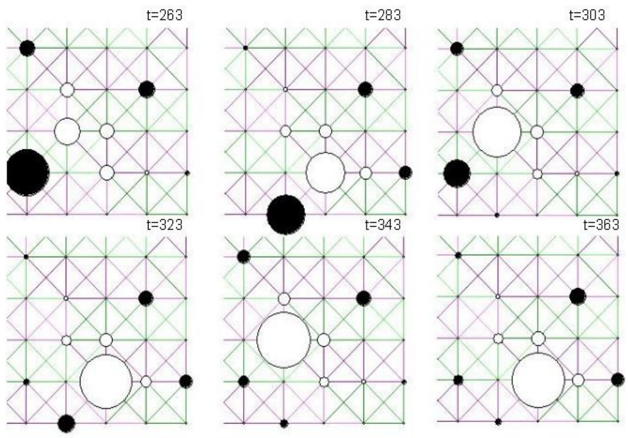

Fig. 5.1 : The remaining non-empty nodes. The period for the system is of 132 generations. This example has been obtained on a regular lattice with p=0.6.

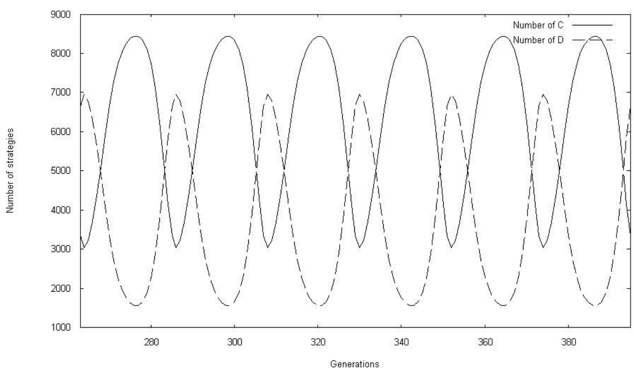

Fig. 5.2 : The evolution of the total populations of C and D strategies over a period of the system. The system is the same as the one considered in Fig. 4.1.

These cases are rare : in Sec. 4, we were able to compute good approximations of the average frequencies of cooperators without taking into account the survival of C-strategies in periodic oscillations. In fact, for Erdös-Rényi random graphs and Barabási and Albert scale-free graphs, the probabilities of periodic evolutions is very low. Table 5.1 gives the probability that a random initialisation gives periodic oscillations.

	Regular lattice	*Random Graph*	*Scale-free*
p=0.4	0.002	0	0
p=0.5	0.005	0.00025	0.00014
p=0.6	0.012	0.002	0
p=0.7	0.008	0	0

Tab. 5.1 : probabilities for periodic evolutions with respect to the topologies

We try to understand the differences between topologies observed in Tab. 5.1 by giving some general features on periodic evolutions. In these evolutions, only some communities remain non-empty, the whole population is concentrated on these communities. The remaining C-communities are clusterized whereas the D-communities are not. In fact, a case of periodic evolution is a case of a C-cluster which is surrounded by a boundary of D-communities. The differences on the probabilities of cyclic evolutions associated with each topology are partially due to the differences in the probabilities of emergence of C-clusters in each topology.

In Tab. 5.2 we give some average information about periodic cases. The average values are computed upon the sets of periodic evolutions we have generated. These sets of periodic evolutions come from massive simulations on graphs of cardinality 100, we ran 10^5 experiments for each kind of graph.

	Regular lattice	*Random graph*	*Scale-Free*
Average number of non-empty C-communities	5.874	9.833	45.5
Average number of non-empty D-communities	5.922	12.833	8
Average number of C-neighbours for a C-community	3.134	2.58	11.54
Average number of D-neighbours for a C-community	1.17	1.39	1.0879
Average number of D-neighbours for a D-community	0.62	0.36	0.375
Average number of C-neighbours for a D-community	1.16	1.065	6.1875

Tab. 5.2 : some general features about periodic cases, each value is computed as an average among the set of periodic cases.

The data of Tab. 5.2 confirm the existence of clusters of C-communities in cases of periodic evolution. For all topologies, a C-community is generally linked to only one D-community : Secs. 3 and 4 insure us that each community is linked with at least one D-neighbour, otherwise we would have an protected community and an evolution towards a cooperative stationary state. Table 5.2 also show that in the case of random graphs and regular lattices, each D-community inhibits in general one C-community. For scale-free graphs, there are very few D communities among the non-empty communities with regard to the number of C-communities, each of the D-communities inhibits a large number of C communities.

A systematic analysis on the periodic cases revealed the existence of a particular sub-structure among the non-empty communities for each case of periodic evolution. This configuration is given in Fig. 5.3.

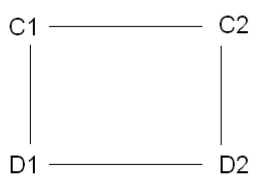

Fig. 5.3 : A pattern of four communities that is always found in a case of periodic evolution. C1 and C2 are C communities, D1 and D2 are D communities.

Here, C1 and C2 are C communities whereas D1 and D2 are D communities. In this configuration, the growth of C1 induces a growth of D1. This growth for the population of D1 corresponds to an increase in the number of contacts for D2 and a decrease in the population of D2. This evolution correspond to an increase of the average score for C2 as C1 growths and D2 decreases. After a certain time, C2 begins to growth and C1 begins to decrease. The situation is then similar to the initial situation we assumed. This configuration is the one inducing periodic evolutions among the remaining non empty communities.

We tested the robustness of the periodic dynamic in our model. For each initial configuration inducing a periodic evolution, we launched the evolution for 500 generations. We then extracted the set of population sizes for non-empty communities for a randomly chosen generation between 100 and 500. For each of these integer values, we modified it by adding a value randomly generated with a N(0,G) law. We didn't allow for communities to be emptied by this mechanism : a new population size inferior to 0 was set to 1. We tested the evolution of the system with the newly obtained set of population sizes. For each value of G, we could compute the frequency of new cases exhibiting periodic evolutions. Results are given by Fig. 5.4 (the global population for all topologies is 10^4) :

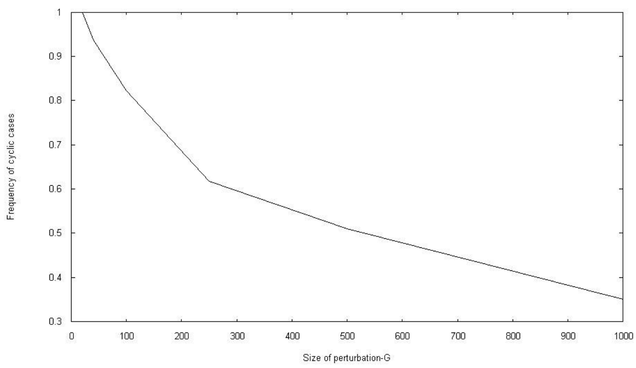

Fig. 5.4 Evolution of the frequency of periodic cases among the modified cases generated with perturbation G.

We noticed that the cyclic phenomenon is robust as periodic evolution is maintained even with major perturbations. Furthermore, the sets of populations values for the new cyclic situations were different from the sets of population values of the cycle induced by the initial configuration.

6. Conclusion and Discussion

We have introduced a new simple mechanism for the emergence of cooperation with basic strategies. In our model the global features we observe come from interactions between a global and a local level. The state of every node of the graph influences the state of every other node in the graph as the population for one particular community is defined using a global average score computed on all the strategies of the network.

The system exhibits three possible evolutions. We have found a criterion to determine the final state from the initialization. If there is an protected community in the initial configuration, the system will converge towards a large majority of cooperators. protected communities correspond to clusterized structures of C-communities : as in the model of Nowak and May, clusters of C-communities are essential to the emergence of cooperation. In our model, the existence of protected communities has a more radical effect than it has in the model of Nowak and May as it nearly induces the disappearance of D strategies. The average frequency of cooperators associated with a topology is therefore directly linked with the probability that an protected community emerges in this topology. Graphs with hubs which are D-communities make it difficult for the emergence of cooperation as these hubs inhibit a large number of C-communities.

In the case of initial states without any protected communities, the network converges towards a all-D population or undertakes periodic evolutions. Clusterized topologies are more likely to induce periodic evolutions as the periodic evolution depends on the existence of a cluster of C strategies which is surrounded by D-communities. The cyclic phenomenon we observe in the model is robust.

The convergence towards cooperation corresponds here to the existence of protected communities. As in other models, emergence of cooperation corresponds to the the fact that cooperative strategies interact between them.

Our model can be seen as a metaphor for a global economy as the evolution of a strategy depends on the average strength among all other strategies which can be seen as a global variable. The evolution towards cooperation can then be seen as the success of communities that benefit from some local protection.

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