JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 75, 236–241 (1980)

A Counterexample Concerning Iteratively Generated Sequences

J. P. R. R. DELAHAYE*

20, rue Lucien Dupuis, 28500 Vernouillet, France

Submitted by R. Bellman

We present a counterexample concerning the cluster set of a sequence (x_n) iteratively generated by a relation of sort $x_{n+1} = f(x_n)$. This counterexample answers a question posed in 1970 by F. T. Metcalf and T. D. Rogers (*J. Math. Anal. Appl.* 31 (1970), 206–212).

1. INTRODUCTION

An article by Metcalf and Rogers [1] concluded with the following problem:

For any given continuous application T of the metric space Y into itself, and for any given point $x_0 \in Y$ is it true that:

$$\mathscr{L}' = T\mathscr{L}' ?$$

$\mathscr{L}$ stands for the set of cluster points of the sequence (x_n) defined by the recursive relation $x_{n+1} = f(x_n)$ from the point x_0.

$\mathscr{L}'$ stands for the derived set of $\mathscr{L}$ (i.e., the set of accumulation points of $\mathscr{L}$).[1]

The answer is negative and we propose in this paper a counterexample.

This counterexample could not be simple, because if $\mathscr{L}' \neq T\mathscr{L}'$ then $\mathscr{L}'$ is infinite [1, Theorem 3]. The inclusion $\mathscr{L}' \subset T\mathscr{L}'$ is always satisfied [1, Lemme, p. 210] so the most simple schema we could hope for $\mathscr{L}'$ was the following:

$$\mathscr{L}' \text{ denumerable,}$$

$$\mathscr{L}' = \{l_1, l_2, ..., l_n, ...\} \cup \{l\}, \qquad \mathscr{L}'' = \{l\},$$

$$T\mathscr{L}' = \{l_0, l_1, l_2, ..., l_n, ...\} \cup \{l\},$$

$$Tl_{n+1} = l_n, \qquad n \in \mathbb{N}^*.$$

* Université des Sciences et Techniques de Lille, Lille, France.

[1] A detailed study of the properties of $\mathscr{L}$ and various other counterexamples concerning those problems are given in [2].

0022-247X/80/050236–06$02.00/0

This was possible and the space Y we are proposing is a subspace of $\mathbb{R}^2$, hence verifying

$$Y' \neq \varnothing, \qquad Y'' \neq \varnothing, \qquad Y''' \neq \varnothing, \qquad Y^{(4)} = \varnothing .$$

2. Description of Y

Let

$$A = \left\{\left(\frac{1}{2^n}, 0\right) \mid n \in \mathbb{N}\right\} \cup \{(0, 0), (3/2, 0)\},$$

$$B = \left\{\left(\frac{1}{2^n}, \frac{1}{2^m}\right) \mid m \in \mathbb{N}, n \in \mathbb{N}, m \geqslant n\right\},$$

$$C = \left\{\left(\frac{3}{2}, \frac{1}{2^n}\right) \mid n \in \mathbb{N}\right\},$$

$$D = \left\{\left(\frac{3}{2}, \frac{1}{2^n} + \frac{1}{2^m}\right) \mid n \in \mathbb{N}^*, m \in \mathbb{N}^*, n < m \leqslant 2n\right\},$$

$$E = \left\{\left(\frac{1}{2^n} + \frac{1}{2^m}, \frac{1}{2^n} + \frac{1}{2^m}\right) \mid n \in \mathbb{N}^*, m \in \mathbb{N}^*, n < m \leqslant 2n\right\}.$$

For any point $X = (x, y)$ in the set $B \cup E$, we take a sequence $(X_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^2$, verifying:

(i) all the X_n introduced in this operation are distinct,

(ii) for any given $X = (x, y) \in B \cup E$ the set $\{X_n \mid n \in \mathbb{N}\}$ has a diameter $\leqslant y/4$, and (X_n) converges to X.

(This is to have the relation

$$\bigcup_{X \in B \cup E} \{X_n \mid n \in \mathbb{N}\}' = B \cup E \cup A - \{(\tfrac{3}{2}, 0)\}.)$$

It is possible to define the X_n, for example, in the following way:

if

$$X = \left(\frac{1}{2^p}, \frac{1}{2^q}\right) \in B,$$

$$X_n = \left(\frac{1}{2^p}, \frac{1}{2^q}\right)_n = \left(\frac{1}{2^p} + \frac{1}{2^{q+2+n}}, \frac{1}{2^q}\right);$$

if

$$X = \left(\frac{1}{2^p} + \frac{1}{2^q}, \frac{1}{2^p} + \frac{1}{2^q}\right) \in E,$$

$$X_n = \left(\frac{1}{2^p} + \frac{1}{2^q}, \frac{1}{2^p} + \frac{1}{2^q}\right)_n = \left(\frac{1}{2^p} + \frac{1}{2^q} + \frac{1}{2^{q+2+n}}, \frac{1}{2^p} + \frac{1}{2^q}\right).$$

Then we propose:

$$Y = A \cup B \cup C \cup D \cup E \cup \bigcup_{X \in B \cup E} \{X_n \mid n \in \mathbb{N}\}.$$

F is closed in $\mathbb{R}^2$, because any convergent sequence of points of Y possesses its limit in F. With the distance of $\mathbb{R}^2$ (e.g., $d((x, y), (x', y')) = ((x - x')^2 + (y - y')^2)^{1/2}$), F is a compact metric space.

3. Description of T

The application T is defined in the following manner:

On A

$$T(0, 0) = (0, 0),$$
$$T(3/2, 0) = (0, 0),$$
$$T(1/2^{m+1}, 0) = (1/2^m, 0), \qquad m \in \mathbb{N},$$
$$T(1, 0) = (3/2, 0).$$

On B

$$T\left(1, \frac{1}{2^m}\right) = \left(\frac{3}{2}, 0\right), \qquad m \in \mathbb{N},$$

$$T\left(\frac{1}{2^n}, \frac{1}{2^m}\right) = \left(\frac{1}{2^{n-1}}, \frac{1}{2^m}\right), \qquad n \in \mathbb{N}^*, \quad m \in \mathbb{N}^*, \quad m \geqslant n,$$

On C

$$T\left(\frac{3}{2}, \frac{1}{2^n}\right) = \left(\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}}, \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}}\right)_0, \qquad n \in \mathbb{N}.$$

On D

$$T\left(\frac{3}{2}, \frac{1}{2^n} + \frac{1}{2^m}\right) = \left(\frac{1}{2^n} + \frac{1}{2^{m+1}}, \frac{1}{2^n} + \frac{1}{2^{m+1}}\right)_0,$$
$$n \in \mathbb{N}^*, \quad m \in \mathbb{N}^*, \quad n < m \leqslant 2n - 1,$$

$$T\left(\frac{3}{2}, \frac{1}{2^n} + \frac{1}{2^{2n}}\right) = \left(\frac{1}{2^n}, \frac{1}{2^n}\right)_0, \qquad n \in \mathbb{N}^*.$$

On E

$$T\left(\frac{1}{2^n} + \frac{1}{2^m}, \frac{1}{2^n} + \frac{1}{2^m}\right) = \left(\frac{1}{2^{n-1}} + \frac{1}{2^{m-1}}, \frac{1}{2^{n-1}} + \frac{1}{2^{m-1}}\right),$$
$$n \in \mathbb{N}^*, \quad m \in \mathbb{N}^*, \quad 1 < n < m \leqslant 2n - 1,$$

$$T\left(\frac{1}{2^n}+\frac{1}{2^{2n}},\frac{1}{2^n}+\frac{1}{2^{2n}}\right)=\left(\frac{1}{2^{n-1}},\frac{1}{2^{n-1}}\right),\qquad n\in\mathbb{N}^*.$$

On $\bigcup_{X\in B}\{X_n \mid n\in\mathbb{N}\}$

$$T\left(1,\frac{1}{2^m}\right)_0=\left(\frac{3}{2},\frac{1}{2^m}\right),\qquad m\in\mathbb{N},$$

$$T\left(1,\frac{1}{2^m}\right)_p=\left(\frac{3}{2},\frac{1}{2^{m+p}}+\frac{1}{2^{2m+p+1}}\right),\qquad m\in\mathbb{N},\quad p\in\mathbb{N}^*,$$

$$T\left(\frac{1}{2^n},\frac{1}{2^m}\right)_p=\left(\frac{1}{2^{n-1}},\frac{1}{2^m}\right)_p,\qquad n\in\mathbb{N}^*,\quad m\in\mathbb{N},\quad m\geqslant n,\quad p\in\mathbb{N}.$$

On $\bigcup_{X\in E}\{X_n \mid n\in\mathbb{N}\}$

$$T\left(\frac{1}{2^n}+\frac{1}{2^{2n}},\frac{1}{2^n}+\frac{1}{2^{2n}}\right)_p=\left(\frac{1}{2^{n-1}},\frac{1}{2^{n-1}}\right)_{p+1},\qquad n\in\mathbb{N}^*,\quad p\in\mathbb{N},$$

$$T\left(\frac{1}{2^n}+\frac{1}{2^m},\frac{1}{2^n}+\frac{1}{2^m}\right)_p=\left(\frac{1}{2^{n-1}}+\frac{1}{2^{m-1}},\frac{1}{2^{n-1}}+\frac{1}{2^{m-1}}\right)_{p+1},$$

$$n\in\mathbb{N}^*,\quad m\in\mathbb{N}^*,\quad 1<n<m\leqslant 2n-1,\quad p\in\mathbb{N}.$$

To verify that T is continuous, we consider all converging sequences $(Z_n)_{n\in\mathbb{N}}$ in Y and we discover that

$$\lim_{m\to\infty} TZ_m = T\lim_{m\to\infty} Z_m .$$

Y is a metric space so this is sufficient to obtain continuity of T.

4. Sequence $(x_n)_{n\in\mathbb{N}}$

The sequence $(x_n)_{n\in\mathbb{N}}$ is then defined from the point $x_0=(1,1)_0$. (See Fig. 1.) T is constructed to have: $\{x_n \mid n\in\mathbb{N}\}=\bigcup_{X\in B\cup E}\{X_n \mid n\in\mathbb{N}\}\cup D\cup C$. So we have:

$$\mathscr{L}=B\cup E\cup A,$$

$$\mathscr{L}'=A-\{(\tfrac{3}{2},0)\},$$

$$T\mathscr{L}'=A.$$

And then: $\mathscr{L}'\neq T\mathscr{L}'$.

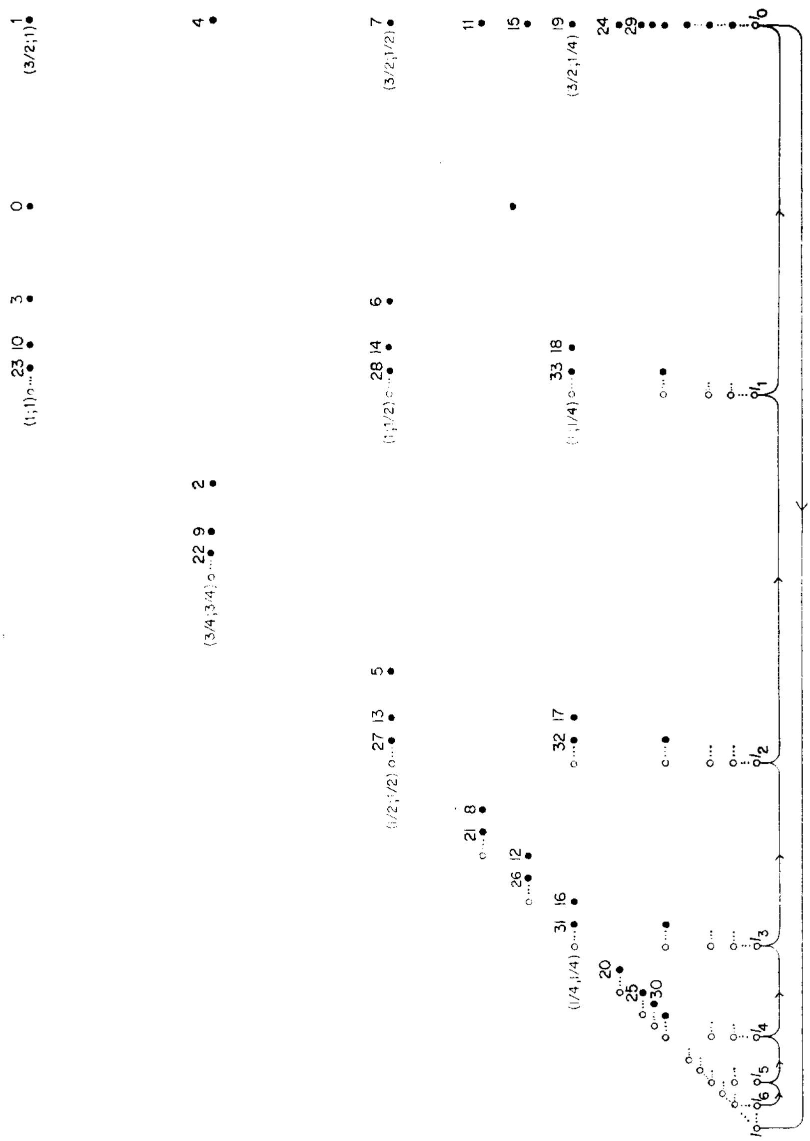

FIG. 1. The points of the sequence (x_n) are represented by solid circles, with the corresponding numbers above. The points of $\mathscr{L}$ are represented by open circles. $T\mathscr{L}' = \{l_0, l_1, ..., l_n, ...\} \cup \{l\}$, $\mathscr{L}' = \{l_1, l_2, ..., l_n, ...\} \cup \{l\}$.

References

1. F. T. Metcalf and T. D. Rogers, The cluster set of sequences of successive approximations, *J. Math. Anal. Appl.* **31** (1970), 206–212.
2. J. P. R. R. Delahaye, "Quelques Problèmes posés par les Suites de Points non Convergentes et Algorithmes pour traiter de Telles Suites," Thèse, Université de Lille, 1979.